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THE SOLUTION OF UNSTEADY COUPLED ELECTROMECHANICAL PROBLEMS FOR MULTILAYERED MEDIA†

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A rigorous formulation of the unsteady coupled electromechanical problem of the interaction of a massive electrode with a multilayered piezoelectric medium is given, and a general formulation of a method of solving it is proposed. As an example the problem for a massive strip electrode, which interacts with a single-layer, double-layer and triple-layer piezoelectric medium with unsteady loading is considered. The effect of the electroelastic properties of different materials (classes 6 mm of the hexagonal system) on the displacement of the electrode and the potential is pointed out.

The majority of investigations of the dynamic processes in electroelastic semi-bounded media are carried out on the assumption that the wave fields vary harmonically with time. However, in practice, we are particularly interested in developing methods for solving such problems in the case of unsteady electric and mechanical loading of piezoelectric media, taking their multilayered form and the electrode mass into account.

1. FORMULATION OF THE PROBLEM

Consider a multilayered piezoelectric semi-bounded medium, occupying the region $-H \leqslant z \leqslant 0$, $-\infty \leqslant x, y \leqslant +\infty$, $H = 2(h_1 + \ldots + h_N)$, where h_i is the half-thickness of the ith layer. Unsteady excitation of the medium occurs via a single electrode of mass m with a plane base S, which is modelled by a rigid punch. There is complete adhesion in the contact area S. The mechanical load, which varies in a specified way with time t, which acts on the electrode, reduced to the centre of mass with coordinates $(0, 0, s)$, is split into a force component $\mathbf{P}(t) = \{P_1, P_2, P_3\}$ and a moment $\mathbf{M}(t) = \{M_1, M_2, M_3\}$. Electric excitation is provided by an electric field $E(t)$ or a current $I(t)$ which vary in a specified way. The system is at rest at the initial instant of time. The displacements of points of the electrode $\mathbf{u}^{(0)}(t) = \{u_1^0, u_2^0, u_3^0\}$ are defined in the form $\mathbf{u}^{(0)} = \mathbf{u} + [\boldsymbol{\varphi} \times \mathbf{r}]$, where $\mathbf{u}(t) = \{u_1, u_2, u_3\}$, u_1, u_2, u_3 are the horizontal and vertical components of the displacement of the centre of mass of the electrode, $\boldsymbol{\varphi} = \{\varphi_1, \varphi_2, \varphi_3\}$ is the vector of the angles of rotation about the centre of mass, and $\mathbf{r} = \{x_1, x_2, -s\}$ is the radius vector of points of the electrode base.

The equations of motion of the solid are described by the equations

$$m\ddot{\mathbf{u}} = \mathbf{P} - \mathbf{Q}, \quad \mathbf{J}\ddot{\boldsymbol{\varphi}} = \mathbf{M} - \mathbf{R}$$

where the vectors $\mathbf{Q}$ and $\mathbf{R}$ are the forces and moments which arise in the contact area between the solid and the medium, $\mathbf{J}$ is a matrix, only the diagonal elements J_1, J_2 and J_3 of which are non-zero, and J_i are the principal moments of inertia about the x_i axes; a dot denotes a partial derivative with respect to time t.

The equations of motion of a multilayered piezoelectric medium [1, 2] include the linear equations of state, the equations of motion in stresses, the equations of electrostatics and the Cauchy relations for each layer

$$\sigma_{ij} = c_{ijkl}s_{kl} - e_{kij}E_k, \quad d_i = e_{ikl}s_{kl} + \varepsilon_{ik}E_k$$

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$$\partial\sigma_{ij} / \partial x_j - \rho\ddot{w}_i = 0, \quad \partial d_i / \partial x_i = 0 \tag{1.1}$$

$$s_{kl} = (\partial w_k / \partial x_1 + \partial w_1 / \partial x_k) / 2, \quad E_k = -\partial\psi / \partial x_k$$

In these equations, for simplicity, we have omitted the superscript $m = 1, 2, \ldots, N$, representing the number of the corresponding layer, σ_{ij}^m are the components of the mechanical-stress tensor, s_{ij}^m are the components of the strain tensor, E_i^m, d_i^m are the components of the electric field and electric induction vectors, w_i^m are the components of the displacement vector, ψ^m is the electric potential, $c_{ijkl}^m = c_{ijkl}^{mE}$ are the components of the elasticity constants vector, measured in a constant electric field, e_{ikl}^m are the components of the piezomoduli tensors, $\varepsilon_{ik}^m = \varepsilon_{ik}^{ms}$ are the components of the permittivity tensor for constant deformations and ρ^m is the density of the material. The properties of the material-constants tensor are described in [1] (i, j, k, l = 1–3, and summation is carried out over repeated subscripts).

Eliminating all the variables apart from w_i and ψ from (1.1), we obtain a system of second-order partial differential equations

$$\begin{aligned} &c_{ijkl}\partial^2 w_k / \partial x_1 \partial x_j + e_{kij}\partial^2\psi / \partial x_k \partial x_j - \rho\ddot{w}_i = 0 \\ &e_{ikl}\partial^2 w_k / \partial x_1 \partial x_i - \varepsilon_{ik}\partial^2\psi / \partial x_k \partial x_i = 0 \end{aligned} \tag{1.2}$$

Henceforth, in addition to numerical indexation of the coordinate axes and vectors, which is necessary for the tensor description, we will also use the traditional notation $\mathbf{x} = \{x_1, x_2, x_3\} = \{x, y, z\}$.

We need to add the initial and boundary conditions of the problem to Eqs (1.2). We will assume the initial values to be zero. The boundary conditions for the mechanical variables are formulated in the same way as the conditions in the problems in the theory of elasticity.

The following contact condition must be satisfied on the surface of the medium $z = 0$

$$\mathbf{w}(x, y, 0, t) = \mathbf{u}^0(x, y, t) = \mathbf{u} + [\boldsymbol{\varphi} \times \mathbf{r}], \quad (x, y) \in S$$

There are no stresses $\mathbf{q} \equiv \{\sigma_{13}, \sigma_{23}, \sigma_{33}\} \equiv \{q_1, q_2, q_3\}$ outside the contact area S

$$\mathbf{q}(x, y, t) = 0, \quad (x, y) \notin S$$

The electrical conditions on the medium surface depend on the type of excitation.

1. When oscillations are excited by an electric field from a voltage generator on a surface electrode connected to it, the known value of the potential

$$\psi = \psi_0(t), \quad (x, y) \in S$$

is specified.

2. If the surface electrode is supplied with a current of known magnitude $I(t)$, the unknown value of the potential $\psi = \psi_0(t)$ is specified on it, and this is determined from the condition

$$\dot{D}_3 = I(t), \quad (x, y) \in S$$

The total charge on the surface of the electrode D_3 is defined in terms of the normal component of the electric-induction vector

$$D_3 = -\iint_s d_3 dS$$

3. If no electrical energy is supplied to the electrode and none is taken from it, the value of $\psi_0(t)$ is found from the condition of conservation of charge

$$\dot{D}_3 = 0, \quad (x, y) \in S$$

There are no free charges on the non-electrode part of the surface, i.e.

$$d_3 = 0, \quad (x, y) \notin S$$

At the interface between the layers the conditions for the mechanical displacements and the electric potential to be equal, and also the conditions for the corresponding components of the elastic stresses and of the normal components of the electric induction to be equal must be satisfied

$$\mathbf{w}^m = \mathbf{w}^{m+1}, \quad \psi^m = \psi^{m+1}, \quad m = 1, 2, \ldots, N-1$$

$$\sigma_{j3}^m = \sigma_{j3}^{m+1}, \quad j = 1-3, \quad d_3^m = d_3^{m+1}, \quad z = -2\sum_{i=1}^{m} h_i$$

For a packet of piezoelectric layers, the lower face of which is clamped and metallized, the following conditions must be satisfied

$$\mathbf{w}^N(x, y, -H, t) = 0, \quad \psi^N(x, y, -H, t) = 0$$

For a multilayered medium, coupled to a half-space, we need to add the conditions for the displacements and the potential to die away as $z \to \infty$

$$\mathbf{w}^N(x, y, z, t) \to 0, \quad \psi^N(x, y, z, t) \to 0$$

2. GOVERNING EQUATIONS

Henceforth we will introduce augmented vectors $\mathbf{w} = \{w_1, w_2, w_3, \psi\}$, $\mathbf{q} = \{q_1, q_2, q_3, d_3\}$, $\mathbf{u}^0 = \{u_1^0, u_2^0, u_3^0, \psi_0\}$. After applying integral Laplace and Fourier transformations with respect to the variables t and x, y and taking the initial and boundary conditions of the problem into account, we can reduce Eqs (1.2) to a system of four integral equations of the first kind

$$\iint_S \mathbf{k}(x-\xi, y-\zeta, p)\mathbf{q}(\xi, \zeta, p)d\xi d\zeta = \mathbf{u}^0(x, y, p), \quad (x, y) \in S \tag{2.1}$$

$$\mathbf{k}(x, y, p) = \frac{1}{4\pi^2} \iint_{\sigma_1\sigma_2} \mathbf{K}(\alpha, \beta, 0, p) e^{-i(\alpha x + \beta y)} d\alpha d\beta$$

with respect to the unknown vector $\mathbf{q}$, where α, β are the parameters of the Fourier transformation and p is the parameter of the Laplace transformation.

The matrix function $\mathbf{K}(\alpha, \beta, z, p)$ is determined by the type of medium, and for multi-layered media has the same form as in the corresponding problems of steady-state oscillations with the oscillation frequency ω replaced by ip.

For a piezoelectric medium of class 6 mm of the hexagonal system, the matrix $\mathbf{K}$ has the structure

$$\mathbf{K} = \begin{Vmatrix} \alpha^2 M_1 + \beta^2 N & \alpha\beta(M_1 - N) & i\alpha M_2 & i\alpha M_3 \\ \alpha\beta(M_1 - N) & \beta^2 M_1 + \alpha^2 N & i\beta M_2 & i\beta M_3 \\ -i\alpha K_1 & -i\beta K_1 & K_2 & K_3 \\ -i\alpha R_1 & -i\beta R_1 & R_2 & R_3 \end{Vmatrix} \tag{2.2}$$

Note that when $z = 0$ we have $K_1 = M_2$, $R_1 = M_3$, $K_3 = R_2$, where M_i, K_i, R_i, N are even functions of the parameters λ, $\omega = ip(\lambda^2 = \alpha^2 + \beta^2)$ and are regular on the real axis everywhere with the exception of the poles $\lambda = \pm p_k$ $(k = 1, 2, \ldots, n)$, which are the same for all types of functions. The contours of integration of σ_1 and σ_2 are chosen in accordance with the principle of limiting absorption [3]. With these properties of the kernels, system (2.1) is uniquely solvable in the space L_p in the region $S(p > 1)$. The criteria of uniqueness are defined in [3, 4].

We will denote by $\mathbf{q}^k$ $\{q_1^k, q_2^k, q_3^k, d_3^k\}$ $(k = 1, \ldots, 7)$ the solutions of the system of integral equations (2.1) for the right-hand side, when only one of the components u_1, u_2, u_3, φ_1 $\varphi_2, \varphi_3, \psi_0$ is non-zero, and has a single value. The corresponding forces and moments, which arise in the contact area between the electrode and the medium, and the total charges on the electrode are given by the following formulae (integration is over the contact area S)

$$R_1^k = \iint (q_3^k x_2 + q_2^k s)dS,\quad R_2^k = \iint (-q_1^k s - q_3^k x_1)dS,\quad R_3^k = \iint (q_2^k x_1 - q_1^k x_2)dS$$
$$Q_i^k = \iint q_i^k dS,\quad i = 1,2,3,\quad D_3^k = \iint d_3^k dS \tag{2.3}$$

Note that q_i^k, Qk_i, R_i^k, d_3^k, D_3^k are functions of the parameter $p = -i\omega$, which is an important feature of the problems considered.

We will write the equations of motion of the punch in Laplace transforms and taking (2.3) into account in the form

$$mp^2 u_i = P_i - \sum_{k=1}^{3} Q_i^k u_k - \sum_{k=4}^{6} Q_i^k \varphi_{k-3} - \psi_0 Q_i^7$$
$$J_i p^2 \varphi_i = M_i - \sum_{k=1}^{3} R_i^k u_k - \sum_{k=4}^{6} R_i^k \varphi_{k-3} - \psi_0 R_i^7,\quad i = 1-3 \tag{2.4}$$

Depending on the type of electric excitation, in Eqs (2.4) ψ is a specified quantity in the case when the electric boundary conditions 1 are used, or is unknown (conditions 2 and 3). In the first case system (2.4) contains six unknowns u_i and φ_i. In the second case, we must add to the equations

$$I(p) = -p\left(\sum_{k=1}^{3} D_3^k u_k + \sum_{k=4}^{6} D_3^k \varphi_{k-3} + \psi_0 D_3^7\right) \tag{2.5}$$

Finally, we obtain a system of seven equations in terms of seven unknowns u_i, φ_i, ψ_0.

3. CONSTRUCTION OF GREEN'S MATRIX FUNCTION FOR A MULTILAYERED MEDIUM

Suppose the medium is a packet of N rigidly coupled electroelastic layers of thickness $H = 2(h_1 + \ldots + h_N)$ with rigidly clamped lower face and occupying the region $H \leqslant z \leqslant 0$, $-\infty \leqslant x, y \leqslant +\infty$

We will introduce a local system of coordinates for each layer

$$z_k = z + 2\sum_{i=1}^{k-1} h_i + h_k,\quad k = 1,2,\ldots,N$$

We will formally separate the layers. Then, the displacement of points of the kth layer w_i^k (i = 1–3) and the electric potential $w_4^k = \psi$ will be given by the expression

$$\mathbf{w}^k(z_k) = \mathbf{B}_+(z_k)\mathbf{Q}_{k-1} + \mathbf{B}_-(z_k)\mathbf{Q}_k,\quad k = 1,2,\ldots,N \tag{3.1}$$

where $\mathbf{Q}_k = \{Q_1, Q_2, Q_3, D_3\}$ is a vector whose components are the forces and electric induction, characterizing the interaction between the layers, and Q_0 is a vector specified on the surface of the medium.

The matrices $\mathbf{B}_\pm(z_k)$ are constructed by the method of eigenvector functions [5] and have a structure of the form (2.2), described in detail in [6]. The elements of these matrices contain elastic, piezoelectric and dielectric moduli of the kth layer and are given for specific types of media in [6–8]. It is preferable to use this representation for each layer (3.1) since there are no growing exponential components in the solution for a multilayered medium, which enables us to investigate media with an arbitrary number of layers, each of which may possess complex physical and mechanical properties.

We will write the conditions for the layers to be joined

$$\mathbf{w}^k(-h_k) = \mathbf{w}^{k+1}(h_{k+1}),\quad k = 1,2,\ldots,N-1 \tag{3.2}$$

and the condition on the lower face of the packet of layers

$$\mathbf{w}^N(-h_N) = 0 \tag{3.3}$$

From (3.2) we have the recurrence relation

$$\mathbf{B}_+(-h_k)\mathbf{Q}_{k-1} + [\mathbf{B}_-(-h_k) - \mathbf{B}_+(h_{k+1})]\mathbf{Q}_k = \mathbf{B}_-(h_{k+1})\mathbf{Q}_{k+1} \tag{3.4}$$

From (3.3) we determine

$$\mathbf{Q}_N = -\mathbf{B}_-^{-1}(-h_N)\mathbf{B}_+(-h_N)\mathbf{Q}_{N-1} \tag{3.5}$$

Using (3.4) and (3.5) we can express $\mathbf{Q}_k$ in terms of the surface load $\mathbf{Q}_0$

$$\mathbf{Q}_k = (-1)^k \prod_{i=k}^{1} \mathbf{F}_i^{-1}\mathbf{B}_+(-h_i)\mathbf{Q}_0, \quad k = 1,2,\ldots,N; \quad \mathbf{F}_N = \mathbf{B}_-(-h_N)$$

$$\mathbf{F}_k = \mathbf{B}_-(-h_k) - \mathbf{B}_+(h_{k+1}) + \mathbf{B}_-(h_{k+1})\mathbf{F}_{k+1}^{-1}\mathbf{B}_+(-h_{k+1}), \quad k = 1,2,\ldots,N-1$$

As a result, the displacements of points of the multilayered medium and the electric potential will be given by the expression

$$\mathbf{w}(z) = \mathbf{K}(\alpha,\beta,z,p)\mathbf{Q}_0, \quad z = z_k - 2\sum_{i=1}^{k-1} h_i - h_k, \quad k = 1,2,\ldots,N$$

$$\mathbf{K}(\alpha,\beta,z,p) = (-1)^{k-1}[\mathbf{B}_+(z_k) - \mathbf{B}_-(z_k)\mathbf{F}_k^{-1}\mathbf{B}_+(-h_k)] \prod_{i=k-1}^{1} \mathbf{F}_i^{-1}\mathbf{B}_+(-h_i)$$

The solution of the problem of a multilayered medium rigidly coupled to a half-space is easily obtained by allowing the thickness of the lower layer to tend to infinity, and replacing here the system of coordinates $z^* = z_N - h_N$. Taking the limit we obtain

$$\mathbf{F}_N = 0, \quad \mathbf{F}_{N-1} = \mathbf{B}_-(-h_{N-1}) - \mathbf{B}_+^\infty(0)$$

$$\mathbf{F}_k = \mathbf{B}_-(-h_k) - \mathbf{B}_+(h_{k+1}) + \mathbf{B}_-(h_{k+1})\mathbf{F}_{k+1}^{-1}\mathbf{B}_+(-h_{k+1}), \quad k = 1,2,\ldots,N-2$$

$$z = z_k - 2\sum_{i=1}^{k-1} h_i - h_k, \quad k = 1,2,\ldots,N-1; \quad z = z^* - 2\sum_{i=1}^{N-1} h_i, \quad k = N$$

4. SOLUTION OF THE CONTACT PROBLEM

To find $\mathbf{u}^0$, φ, ψ_0 we need to find the functionals $\mathbf{R}^k$, $\mathbf{Q}^k$, D_3^k, which are related to the fundamental solutions of the system of integral equations (2.1) $\mathbf{q}^k$ by relations (2.3). The solutions $\mathbf{q}^k = \{q_1, q_2, q_3, d_3\}$ are constructed by the method of fictitious absorption, which enables us to take into account analytically all the singularities on the boundary of the contact area S.

According to the method of fictitious absorption, the solution of the system of integral equations (2.1) $\mathbf{Kq} = \mathbf{f}$ will be sought in the form

$$\mathbf{q}(x,y) = \mathbf{p}_0(x,y) + \mathbf{p}_*(x,y) \tag{4.1}$$

The unknown function $\mathbf{p}_*(x,y)$ is chosen from the condition for the following functionals to be equal

$$V\mathbf{q}(\pm\alpha_m, \pm\beta_l) = V\mathbf{p}_0(\pm\alpha_m, \pm\beta_l), \quad p_m^2 = \alpha_m^2 + \beta_l^2, \quad m,l = 1,2,\ldots,n$$

Here

$$V(\alpha,\beta)\mathbf{f}(x,y) = \iint_{-\infty}^{+\infty} \mathbf{f}(x,y)e^{i(\alpha x+\beta y)}dxdy$$

$$V^{-1}(x,y)\mathbf{F}(\alpha,\beta) = \int_{-\infty}^{+\infty}\int \mathbf{F}(\alpha,\beta)e^{-i(\alpha x+\beta y)}d\alpha d\beta$$

and $\pm p_m$ is a pole, which is the same for all the elements of the matrix $\mathbf{K}(\alpha,\beta)$.

The vector function $\mathbf{p}_*(x,y)$ contains, by its construction, a certain arbitrariness, which can be eliminated in the final solution of the whole problem [4].

To fix our ideas, suppose S is a rectangle with sides $2a$ and $2b$ ($|x| \leqslant a$, $|y| \leqslant b$). We can then take as $\mathbf{p}_*(x,y)$ the Dirac delta function with carriers at the points x_i and y_i

$$\mathbf{p}_*(x,y) = \sum_{i=1}^{2n}\sum_{j=1}^{2n} \mathbf{C}_{ij}\delta(x-x_i, y-y_j)$$

where $\mathbf{C}_{ij} = \{c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4\}$ are unknown constants to be determined, and $x_i = \pm x_i^0$, $y_i = \pm y_i^0$, x_i^0, y_i^0 are points which divide the region S_0 ($0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$) into equal rectangles.

We will represent the matrix $\mathbf{K}(\alpha, \beta)$ in the form

$$\mathbf{K}(\alpha,\beta) = \mathbf{S}(\alpha,\beta)\,\mathbf{\Pi}(\alpha,\beta);\ \mathbf{\Pi}(\alpha,\beta) = \mathbf{E} + \mathbf{\Pi}_0(\alpha,\beta),\ \alpha^2+\beta^2 = \lambda^2 \tag{4.2}$$

where $\mathbf{E}$ is the identity matrix, $\mathbf{\Pi}(\alpha, \beta)$ is a matrix whose elements contain all the singularities of the elements of the matrix $\mathbf{K}(\alpha, \beta)$, and $\mathbf{S}(\alpha, \beta)$ has as its elements functions which decrease at infinity as $\lambda \to \infty$, and which contain no singularities on the real axis [4].

We will introduce a new unknown vector, namely, the function $\mathbf{t}(x, y)$, by the relation

$$V(\alpha,\beta)\mathbf{t}(x,y) = \Pi(\alpha,\beta)V(\alpha,\beta)\mathbf{p}_0(x,y) \tag{4.3}$$

which corresponds to

$$\mathbf{T}(\alpha,\beta) = \mathbf{\Pi}(\alpha,\beta)\mathbf{P}_0(\alpha,\beta)$$

As a result, the system of integral equations (2.1), after substituting (4.1) and (4.2) and taking (4.3) into account, can be transformed to the form

$$\mathbf{K}_0\mathbf{t} = \iint_s \mathbf{k}_0(x-\xi, y-\zeta)\mathbf{t}(\xi,\zeta)d\xi d\zeta = \mathbf{g}(x,y),\quad (x,y)\in S \tag{4.4}$$

$$\mathbf{k}_0(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{S}(\alpha,\beta)e^{-i(\alpha x+\beta y)}d\alpha d\beta$$

$$\mathbf{g}(x,y) = \mathbf{f}(x,y) - \frac{1}{4\pi^2}\iint_s \mathbf{k}(x-\xi, y-\zeta)\mathbf{p}_*(\xi,\zeta)d\xi d\zeta$$

For a rectangular contact area

$$\mathbf{g}(x,y) = \mathbf{f}(x,y) - \sum_{i=1}^{2n}\sum_{j=1}^{2n}\mathbf{k}(x-x_i, y-y_j)\mathbf{C}_{ij}$$

The most important part of the method of fictitious absorption is the construction of the operator K_0, which describes the behaviour of waves in media with strong absorption, or which arise when solving static-type problems, since $\mathbf{S}(\alpha, \beta)$ contains no singularities on the real axis and decreases power-wise at infinity.

Without loss of generality we can put $\mathbf{f}(x,y) = \mathbf{A}e^{-i\eta_1 x - i\eta_2 y}$, where η_1, η_2 are constants, and $\mathbf{A} = \{A_1, A_2, A_3, A_4\}$, and we can suppose that $\mathbf{t}_\eta(x,y) = \chi_\eta(x,y)\mathbf{A}$ is the solution $\mathbf{K}_0\mathbf{t} = \mathbf{f}$ constructed by one of the numerous methods for solving problems of statics or for media with strong attenuation (the factorization method, asymptotic methods, methods of orthogonal polynomials, etc. [3, 4, 9–11]). Then, using the superposition principle it can be shown that the solution of (2.1) $\mathbf{Kq} = \mathbf{f}$ is the vector function

$$\mathbf{q}(x,y) = \left\{\chi_\eta(x,y) + \frac{1}{4\pi^2}\int_{\sigma_1}\int_{\sigma_2}(\mathbf{\Pi}^{-1}(\alpha,\beta) - \mathbf{E})\mathbf{X}_\eta(\alpha,\beta)e^{-i(\alpha x+\beta y)}d\alpha d\beta\right\}\mathbf{A} -$$

$$-\frac{1}{4\pi^2}\sum_{i=1}^{2n}\sum_{j=1}^{2n}\left\{\int_{\sigma_1}\int_{\sigma_2}\mathbf{\Pi}^{-1}(\alpha,\beta)\mathbf{L}_{ij}(\alpha,\beta)e^{-i(\alpha x+\beta y)} + \right.$$

$$\left. +(\mathbf{\Pi}^{-1}(\alpha,\beta) - \mathbf{E})e^{-i(\alpha(x-x_i)+\beta(y-y_j))}d\alpha d\beta\right\}\mathbf{C}_{ij} \tag{4.5}$$

$$\mathbf{L}_{ij}(\alpha,\beta)=\frac{\mathbf{S}^{-1}(\alpha,\beta)}{4\pi^2}\int\limits_{\sigma_1\sigma_2}\int(\mathbf{K}(\eta_1,\eta_2)-\mathbf{S}(\eta_1,\eta_2))\mathbf{S}(\alpha,\beta)\mathbf{X}_\eta(\alpha,\beta)e^{i(\eta_1x_i+\eta_2y_j)}d\eta_1d\eta_2 \tag{4.6}$$

Correspondingly

$$\mathbf{Q}(\alpha,\beta)=V\mathbf{q}(x,y)=\mathbf{\Pi}^{-1}(\alpha,\beta)\left\{\mathbf{X}_\eta(\alpha,\beta)\mathbf{A}-\sum_{i=1}^{2n}\sum_{j=1}^{2n}\mathbf{L}^*_{ij}(\alpha,\beta)\mathbf{C}_{ij}\right\} \tag{4.7}$$

$$\mathbf{L}^*_{ij}(\alpha,\beta)=\mathbf{L}_{ij}(\alpha,\beta)-(\mathbf{\Pi}(\alpha,\beta)-\mathbf{E})e^{i(\alpha x_i+\beta y_j)},\quad \eta_1^2+\eta_2^2=\eta^2$$

Note that $\mathbf{L}^*_{ij}(\alpha,\beta)$ and the integrals in solution (4.5), (4.6) are calculated using the theory of residues and the formulae of the operational calculus.

It follows from (4.3) that

$$\mathbf{p}_0=V^{-1}(x,y)\mathbf{\Pi}^{-1}(\alpha,\beta)\mathbf{T}(\alpha,\beta)$$

In order that the vector function $\mathbf{p}_0$ should be from L_p and only have a carrier in the region S, by the method of fictitious absorption the following relations must be satisfied

$$\mathbf{T}(\pm\alpha_k,\pm\beta_m)=0,\quad \alpha_k^2+\beta_m^2=z_k^2,\quad m,k=1,2,\ldots,n$$

where $\pm z_k$ is a pole of the inverse matrix $\mathbf{\Pi}^{-1}(\alpha,\beta)$, which is the same for all elements of the matrix.

As a result we have the following system of algebraic equations for determining the vector $\mathbf{C}_{ij}$

$$\sum_{i=1}^{2n}\sum_{j=1}^{2n}\mathbf{L}^*_{ij}(\alpha_k,\beta_m)\mathbf{C}_{ij}=\mathbf{T}_\eta(\alpha_k,\beta_m),\quad \alpha_k=\pm\sqrt{z_k^2-\beta_m^2}$$

To construct an approximate solution we only need to satisfy the relations for discrete values of $\beta_m=\pm m$.

5. THE FINAL SOLUTION OF THE PROBLEM

The functionals $\mathbf{Q}^k$, $\mathbf{R}^k$, D_3^k in (2.4) and (2.5) are related to the solution (4.7) of the equation $K\mathbf{q}=\mathbf{A}e^{-\eta_1x-i\eta_2y}$ as follows:

$$\mathbf{Q}^1=\mathbf{Q},\quad \mathbf{A}=\{1,0,0,0\},\quad \mathbf{Q}^2=\mathbf{Q},\quad \mathbf{A}=\{0,1,0,0\}$$

$$\mathbf{Q}^3=\mathbf{Q},\quad \mathbf{A}=\{0,0,1,0\},\quad \mathbf{Q}^7=\mathbf{Q},\quad \mathbf{A}=\{0,0,0,1\}$$

$$\mathbf{Q}^4=i\frac{\partial\mathbf{Q}^3}{\partial\eta_2}+s\mathbf{Q}^2,\quad \mathbf{Q}^5=-i\frac{\partial\mathbf{Q}^3}{\partial\eta_1}-s\mathbf{Q}^1,\quad \mathbf{Q}^6=-i\frac{\partial\mathbf{Q}^1}{\partial\eta_2}+i\frac{\partial\mathbf{Q}^2}{\partial\eta_1}$$

$$R_1^k=-i\frac{\partial Q_3^k}{\partial\beta}+sQ_2^k,\quad R_2^k=i\frac{\partial Q_3^k}{\partial\alpha}-sQ_1^k,\quad R_3^k=-i\frac{\partial Q_2^k}{\partial\alpha}+i\frac{\partial Q_1^k}{\partial\beta}$$

$k=1,\ldots,7$

Everywhere here we must put $\alpha=\beta=\eta_1=\eta_2=0$ in the formulae $\mathbf{Q}(\alpha,\beta)\equiv\mathbf{Q}(\alpha,\beta,\eta_1,\eta_2)$.

After determining the functionals, we obtain the displacements of the centre of mass $u_i(p)$, the angles of rotation of the electrode $\varphi_i(p)$ and the electric potential $\psi_0(p)$ from system (2.4)–(2.5).

The reaction of the base $Q_i(p)$, the moments $R_i(p)$ and the stresses $q_i(x,y,p)$ in the contact area S, the total charge $D_3(p)$ and the electric induction $d_3(x,y,p)$ will be given by the expressions

$$f(p)=\sum_{k=1}^{3}f^ku_k+\sum_{k=4}^{6}f^k\varphi_{k-3}+\psi_0f^7,\quad f=f(q_i,Q_i,R_i,d_3,D_3)$$

To determine the displacements of points of the medium w_i and the potential ψ we need to use the expression

$$\mathbf{w}(x, y, z, p) = \frac{1}{4\pi^2} \int_{\sigma_1} \int_{\sigma_2} \mathbf{K}(\alpha, \beta, z, p)\mathbf{Q}(\alpha, \beta, p)e^{-i(\alpha x+\beta y)}d\alpha d\beta$$

$$\mathbf{w} = \{w_1, w_2, w_3, \psi\}$$

Using the method of fictitious absorption these integrals can be reduced to the form

$$\mathbf{w}(x, y, z, p) = -\sum_{m=1}^{n}\sum_{l=1}^{n} \operatorname*{Res}_{\substack{\operatorname{Im}\alpha_m\leqslant 0\\ \operatorname{Im}\beta_l\leqslant 0}}[\mathbf{K}(\alpha_m, \beta_l, z, p)]\mathbf{P}_*(\alpha_m, \beta_l, p)e^{-i(\alpha_m x+\beta_l y)}$$

For a rectangular contact area we have

$$\mathbf{w}(x, y, z, p) = -\sum_{m=1}^{n}\sum_{l=1}^{n} \operatorname*{Res}_{\substack{\operatorname{Im}\alpha_m\leqslant 0\\ \operatorname{Im}\beta_l\leqslant 0}}[\mathbf{K}(\alpha_m, \beta_l, z, p)]\sum_{i=1}^{2n}\sum_{j=1}^{2n}\mathbf{C}_{ij}e^{-i(\alpha_m(x-x_i)+\beta_l(y-y_j))}, \quad x>a, \ y>b$$

To obtain the final solution in the time t we need to carry out an inverse Laplace transformation. By making the replacement $p = -i\omega$ the corresponding integral can be reduced to a Fourier integral

$$F(t) = \frac{2}{\pi}\int_0^\infty \operatorname{Re} f(i\omega)\cos\omega t d\omega = -\frac{2}{\pi}\int_0^\infty \operatorname{Im} f(i\omega)\sin\omega t d\omega \tag{5.1}$$

The modified Filon method [9] can be used to evaluate integrals of the form (5.1)

6. SOLUTION OF THE PLANE PROBLEM

As an example we will consider the case when the electrode is modelled by a strip-shaped punch of width $2a$. Without loss of generality we can assume that the centre of mass of the punch coincides with the origin of coordinates, i.e. $s = 0$. The system of integral equations (2.1) in the plane formulation has the form

$$\mathbf{Kq} = \mathbf{f} = \begin{Vmatrix} u_1 \\ u_2 - x\varphi \\ \psi_0 \end{Vmatrix}, \quad \mathbf{q} = \begin{Vmatrix} q_1 \\ q_2 \\ d \end{Vmatrix}$$

Suppose q^k are the solutions

$$\mathbf{Kq}^k = \begin{Vmatrix} 1 \\ 0 \\ 0 \end{Vmatrix}, \begin{Vmatrix} 0 \\ 1 \\ 0 \end{Vmatrix}, \begin{Vmatrix} 0 \\ -x \\ 0 \end{Vmatrix}, \begin{Vmatrix} 0 \\ 0 \\ 1 \end{Vmatrix}, \quad k = 1, 2, 3, 4$$

For electric boundary conditions of the type 1, ψ_0 is a specified function. Taking into account the symmetry of the functions, system (2.4) can be written in the form

$$S_1^1 u_1 = P_1 - \varphi Q_1^3, \quad S_2^2 u_2 = P_2 - \psi_0 Q_2^4, \quad R^3\varphi = M - u_1 Q_1^3 \tag{6.1}$$

$$S_i^i = Q_i^i + mp^2, \quad Q_i^k = \int_{-a}^{a} q_i^k(x)dx, \quad R^3 = \int_{-a}^{a} q_2^3(x)dx + Jp^2$$

$$i = 1, 2; \quad k = 1, 2, 3, 4$$

The displacements and angle of rotation are given by the expressions

$$u_1 = \frac{P_1R^3 - Q_1^3M}{\Delta_1}, \quad u_2 = \frac{P_2 - \psi_0 Q_2^4}{S_2^2}, \quad \varphi = \frac{MS_1^1 - Q_1^3P_1}{\Delta_1} \tag{6.2}$$

$$\Delta_1 = S_1^1R^3 - (Q_1^3)^2$$

For electric boundary conditions of type 2 and 3, $\psi_0(p)$ is an unknown function. We add Eq. (2.5) to Eqs (6.1). Taking into account the evenness of the functions d^k we have

$$-p(u_2D^2 + \psi D^4) = I(p), \quad D^k = \int_{-a}^{a} d^k dx, \quad k = 2, 4 \tag{6.3}$$

The solution of system (6.1), (6.3) has the following form: the expressions for u_1 and φ are identical with those in (6.2), while

$$u_2 = -\frac{pD^4P_2 + IQ_2^4}{\Delta_2}, \quad \psi_0 = \frac{IS_2^2 + pD^2P_2}{\Delta_2}; \quad \Delta_2 = -p[D^4S_2^2 - D^2Q_2^4] \tag{6.4}$$

The solutions constructed above enable us to investigate coupled electromechanical problems in the total volume, taking different factors into account: the mass and moments of inertia of the electrode, the coupling in the contact area between the electrode and the underlying medium, and different types of unsteady electrical and mechanical actions on the system. We also note that the use of natural forms of oscillations of solids of finite dimensions in combination with the proposed approach to solving unsteady problems for rigid electrodes enables one to consider the interaction of flexible electrodes with multilayered bases with different types of boundary conditions.

7. NUMERICAL ANALYSIS

A numerical analysis was carried for a massive strip electrode of width $2a$, subjected to the action of a mechanical load of the form $\mathbf{P}(t) = \{0, P(t)\}$ applied at the centre of mass (0, 0) and for two types of electric action: (a) for a specified value of the potential $\psi = \psi_0$, and (b) when a voltage generator with a specified value of the current $I(t)$ is connected to the electrode.

We will dwell on the results which show the effect of the electroelastic properties of the materials on the behaviour of a free electrode (there is no external circuit, i.e. $I(t) = 0$), which interacts with a double-layer and triple-layer medium.

In Fig. 1 we show graphs of the vertical displacements, while in Fig. 2 we show graphs of the potential for the action of a load of the form $P(t) = H(t) - H(t - 0, 1)$ ($H(t)$ is the Heaviside function) on an electrode of unit mass, which interacts without friction with a double-layer medium having a thickness $H = 2$ ($h_1 = h_2 = 1/2$). The lower face of the packet is clamped and metallized. TsTS-19 piezoelectric ceramics [12] is used as the base. The piezoelectric coefficient e_{33} was varied in the lower layer. Curves 1–5 correspond to the values $e_{33}^2 = 0, e_{33}^1/2, e_{33}^1, 3e_{33}^1/2, 2e_{33}^1$ (curve 3 corresponds to a uniform layer of thickness $H = 2$). Since the properties of the upper layer do not change, curves 1–5 are identical until a wave, reflected from the interface of the media, arrives. An increase in the value of the parameter e_{33}^2 of the lower layer with respect to the upper layer leads to a reduction in the period of natural oscillations of the system after the load is removed. The mechanical "stiffness" of the system increases

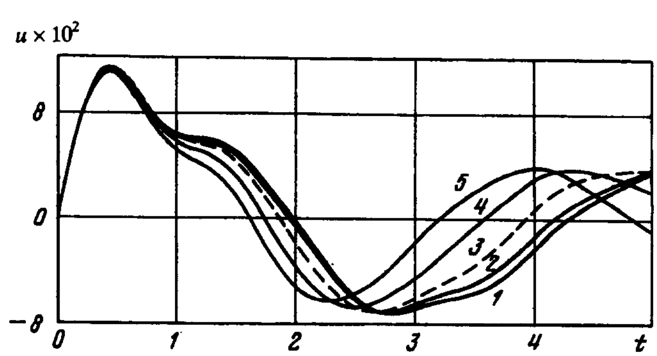

Fig. 1.

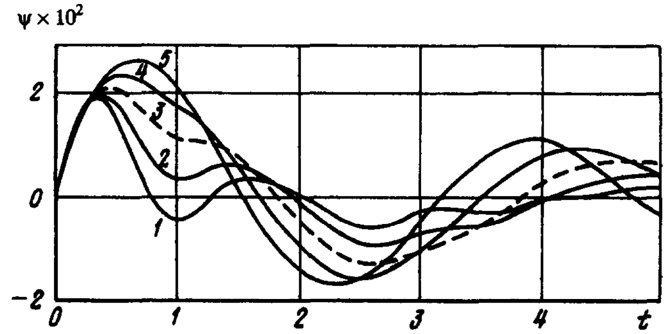

Fig. 2.

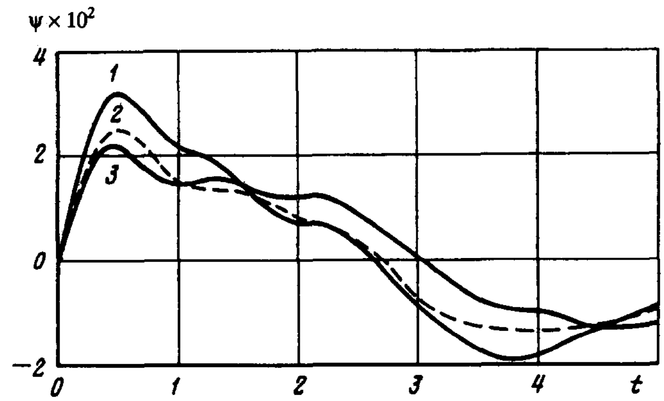

Fig. 3.

as e_{33}^2 increases. It can be seen from Fig. 2 that as the value of e_{33}^2 of the lower layer increases the amplitude ψ increases and the nature of the variation of the electric field changes.

In Fig. 3 we show graphs of $\psi(t)$ when an electrode of unit mass interacts with a triple-layer medium $H = 3$ ($h_1 = h_2 = h_3 = 1/2$), $P(t) = H(t) - H(t - 0, 1)$, $I(t) = 0$. All the parameters of the layers correspond to those of TsTS-19 piezoelectric ceramics, with the exception of the coefficient c_{33}, which was varied in the upper and lower layer with respect to the value in the middle layer. Curve 1 corresponds to a packet with values $c_{33}/2$, c_{33}, $3c_{33}/2$ (which gradually increased the stiffness with depth), curve 2 corresponds to a uniform layer and curve 3 corresponds to the values $3c_{33}/2$, c_{33}, $c_{33}/2$ (which gradually reduced the stiffness with depth). It can be seen that the ratio of the stiffnesses of two neighbouring layers is decisive in influencing the way the potential changes, while it is the stiffness of the upper layer that is decisive for the packet as a whole. The qualitative pattern of the change in the electrode displacements with time is similar to the graphs shown in Fig. 3.

Curve 3 in Fig. 4 corresponds to vertical displacements of a short-circuited electrode (the external electric circuit is closed), which is in contact with a layer of PZT5H, the lower face of which is clamped and metallized. Curves 1, 2, 4 and 5 show the effect of a change in the parameters e_{33} and ε_{33} (see Table 1) on the amplitude and period of oscillations after the load is removed. Fig. 4 illustrates the fact that the effect of the coefficient e_{33} on the displacement of the electrode is less important than the effect of ε_{33}. A change in the electromechanical coupling coefficient $k^2 = \varkappa/(1 + \varkappa)$, $\varkappa = e^2_{33}/(c_{33}\varepsilon_{33})$ due to a change in the coefficient ε_{33} (curves 1 and 5) leads not only

Table 1

Coefficient	Curve number				
	1	2	3 (PZT5H)	4	5
e^{22}, C/m^2	23.3	17,6	23,3	30,3	23,3
ε^{22}, F/m	2.26	1,3	1,3	1,3	0,772
k	0,412	0,412	0,513	0,612	0,612

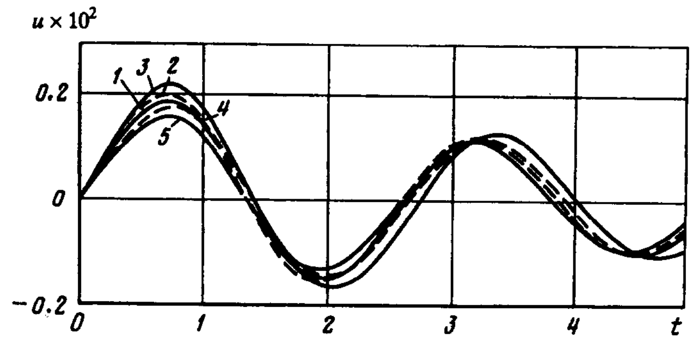

Fig. 4.

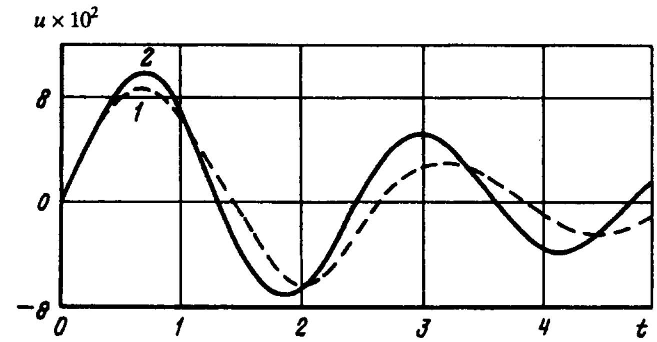

Fig. 5.

to considerable changes in the amplitude, but also to a change in the period of oscillations of the system. A similar picture is observed in "weak" piezoelectrics but in a less pronounced form.

Figure 5 illustrates the behaviour of a massive punch which is in contact with two different media. Curve 1 corresponds to PZT5H piezoelectric ceramics, and curve 2 corresponds to a transversely isotropic medium with elastic parameters corresponding to PZT5H. The differences in the nature of the displacements of the punch when the medium has no piezoelectric properties can be clearly seen. The presence of piezoelectric properties reduces the amplitude and increases the period of the system oscillations after the load is removed. In Figs 4 and 5 the load $P(t) = te^{-10t}$, $m = 1, H = 2h = 1, \psi_0 = 0$.

Calculations were carried out in dimensionless form for viscoelastic media [12]. In this case $\omega = ipe^{-is}$, and ζ is the viscosity parameter of the medium $0 \leqslant 2\zeta \leqslant 1$ (the elasticity constants are complex quantities of the form $c_{ij}e^{2i\zeta}$). In Figs 1–5 $\zeta = 0.2$. The displacements are referred to a, the potential is referred to al ($l = 10^{10}$ and has the dimension of electric field), and the time is referred to a/V (V is the velocity of propagation of transverse waves in the upper layer).

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